Quantum Optics P572 Teaser Question Feb. 3, 2015

Question: *Church of the larger Hilbert space.*

Given a total system-environment state ρ that is an operator on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$, we saw that one can construct a density operator ρ_S on the subsystem S that reproduces all of the observables that act only on S: $\rho_S = \text{Tr}_E \rho$ so that $\mathcal{O} = \text{Tr}_E [\rho_S \mathcal{O}_S]$.

We want a converse: given *any* density matrix ρ acting on a Hilbert space $\mathcal{H}$, we want a pure state $|\psi\rangle$ on a fictitious larger Hilbert space $\mathcal{H}' = \mathcal{H} \otimes \mathcal{A}$ for some "ancilla" $\mathcal{A}$ that reproduces all observables $\mathcal{O}$. That is, for any $\rho \in \mathcal{H}$ there exists a $|\psi\rangle \in \mathcal{H}'$ such that $\langle\psi|\mathcal{O}|\psi\rangle = \text{Tr}\,[\mathcal{O}\rho]$, tracing over the original Hilbert space $\mathcal{H}$.

What is a $|\psi\rangle$ that gives the desired properties? Is the choice of $|\psi\rangle$ unique? What is the minimal dimension required for the larger Hilbert space for this construction in general?

This converse is a useful way to pretend that mixed systems are closed. In a similar way, one can lift mixed state dynamical evolutions, even an irreversible ones (e.g. dissipative master equations that we will learn about), to unitary evolution on a larger Hilbert space.