Problem 1. Homodyne and heterodyne detection

Photodetectors count photons hitting them. The average number of counts $\langle n\rangle$ is proportional to the field intensity, and higher moments of n the counts measure intensity fluctuations. We have also seen how to use interferometers and photodetectors to measure coherences such as $g^{(1)}(\mathbf{r},\tau)$. However, we often want to measure the electric field $\langle E(\mathbf{r},t)\rangle$, not just the intensity $|\langle E(\mathbf{r},t)\rangle|^2$ or the coherence $\langle E^*(\mathbf{r},t)E(\mathbf{r}',t')\rangle$.

This question considers how to measure the electric field in a single mode j (an index we will suppress since there is only one mode), and arbitrary quadratures of it $X_\beta = \beta^* a + \beta a^\dagger$ for some quadrature β, using photodectors and a beam splitter described by (as usual)

$$\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} iR & T \\ T & iR \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1}$$

where R and T are reflection and transmission amplitudes, a_1 and a_2 are the input fields, and a_3 and a_4 are the output fields. For a typical beam splitter, with the i's introduced above the coefficients R and T are real; please assume that for this problem.

(a) **Homodyne/heterodyne detection.** The basic setup to measure the field quadrature X_β for a mode a is a "homodyne measurement": we input mode a as the input 1 to the beam splitter. We also input into mode 2 a *local oscillator* (LO) field that we prepare in a coherent state $|\alpha\rangle$ (the experimenter can control α).

In ordinary homodyne detection, we take $T \ll R$. Calculate the output $\langle n_4\rangle$ measured on a photodector on output port 4 to linear order in T. You should find that the result is the sum of two terms: one that depends only on the LO intensity and one that is proportional to T times a quadrature of a. For LO state $|\alpha\rangle$, what quadrature – i.e. what β in X_β – is obtained by this type of measurement?

(b) **Balanced homodyne detection.** Another type of homodyne detection with some advantages is balanced homodyne detection, in which one uses a balanced beam splitter ($R = T = 1/2$) and measures the output intensities I_3 and I_4; after obtaining these, one calculates the difference of them. Calculate $\langle n_4 - n_3\rangle$ and relate it to the input field quadrature. *Hint: you should see that the only contribution is proportional to the input field, and that there is no term involving only the LO.*

Problem 2. Generating quantum light with nonlinear optics

This problem explores using the nonlinearity of atoms or materials to generate quantum states of light. We concentrate on the simplest building blocks of nonlinear phenomena: low-order nonlinearities and simple configurations. These examples are also some of the most important for optics technologies used in physics labs.

In class, we considered squeezing from Hamiltonian terms like

$$\bar{\xi} a^2 - \bar{\xi}^* (a^\dagger)^2. \tag{2}$$

The notes show how this arises from a $\chi^{(2)}$ nonlinearity (a point which we quickly brushed over in class). To summarize the notes, it turns out that this actually came from a Hamiltonian involving two modes of an electromagnetic field with a coupling $\left(g a_0^\dagger a_1^2 + \text{H.c.}\right)$ where the mode a_0 is in a coherent state, and thus we can replace a_0 with a classical number; thus the Hamiltonian reduces to Eq. (**??**).

(a) **Parametric down conversion.** The above process is basically a special case of "parametric down-conversion", where a photon of frequency ω (the energy of mode 0) converts to two photons of frequency ω_1 and ω_2 (the energies of modes 1 and 2). Consider the more general Hamiltonian describing this

$$H = g\left(a_1^\dagger a_2^\dagger a_0 + \text{H.c.}\right). \quad (3)$$

Assuming the pump field is intense and coherent enough that it is in a constant coherent state, so that a_0 can be replaced with a c-number. Then solve Heisenberg's equation of motion for a_1 and a_2. (*Hint: you will have to solve a coupled set of two operator differential equations. It is a special case (2 modes) of that you solved on the midterm. The final solution will have features reminiscent of the squeezed state obtained in class in the last lecture (hyperbolic cosines, etc.).* Using these solutions, show that if the system is initially in the vacuum (for modes 1 and 2), then at finite times the expectations $\langle n_1 \rangle = \langle n_2 \rangle$ are finite, but the variance in $n_1 - n_2$ vanishes. (This takes a bit of operator algebra. See me if you need help!) Thus, remarkably, one can have a noise-free quantity in some combination of the two variables. This is called a "two-mode squeezed vacuum".

(b) **Finding $\chi^{(2)}$: often zero.** In materials with inversion symmetry, $\chi^{(2)}$ vanishes. Argue that inversion symmetry implies $\chi^{(2)} = 0$ – you can argue it just by using how E and the crystal transform under inversion. For such materials, $\chi^{(3)}$ gives the leading nonlinearity – a *Kerr nonlinearity*. Such nonlinearities can generate interesting non-trivial states, but we will not explore them further.

(c) **Finding $\chi^{(2)}$: engineering with atoms.** For two level atoms in their steady state, calculate $\chi^{(2)}$. *Hint: you should be able to read this off directly from the expressions we derived in class.*