Problem 1. Beam splitter with polarization: from indistinguishable to classical

In class we described how classical particles input on a beam splitter interfere, as well as how indistinguishable particles in Fock states input onto a beam splitter interfere. Here you will study some more general states that can be partially distinguishable. Recall the input/output relation

$$\begin{pmatrix} b^3_{L,\sigma} \\ b^4_{R,\sigma} \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} b^1_{L,\sigma} \\ b^2_{R,\sigma} \end{pmatrix} \tag{1}$$

This is the same as the teaser, except we now have allowed for a light polarization $\sigma \in \{\pm 1\}$ and I have simplified the phase convention. The modes 1 and 2 are input modes, while 3 and 4 are the outputs. Note that the beam splitter's effect doesn't depend on polarization.

(a) In class, we studied input Fock states with a well-defined polarization: $|1, 0; 0, 0\rangle$ for example, where $|nm; ab\rangle$ denotes n particles on the left input mode with $+$ polarization, m with $-$, a on the right input mode with $+$ polarization, and b on the right with $-$. What is the output state if one puts in a single photon in the first mode with a linear polarization (i.e. $(1/\sqrt{2})(|1, 0; 0, 0\rangle + |0, 1; 0, 0\rangle)$?

(b) What is the output state if one puts exactly one photon in each mode with identical, but *arbitrary*, polarization? Please do the algebra for an arbitrary polarization. *This should agree with the single polarization result obtained in class.*

(c) What is the output state if the photon in input mode one is polarized as $a\,|+\rangle + b\,|-\rangle$, but the second input mode is polarized as $a'\,|+\rangle + b'\,|-\rangle$? At some point the output should look like distinguishable, classical particles – when does this happen? *Hint: if $a' = a$ and $b' = b$, this is just the previous question, so check against that case! Also, the point at which things look distinguishable should be intuitive, so check against that.*

Problem 2. Spectral properties and temporal correlation

We defined $g^{(1)}(x_1, x_2)$ in class. For (spatially and temporally) homogeneous cases, this depends only on $x_1 - x_2$. Let's consider correlations at the same point in space, so we have $g^{(1)}(\tau)$ where $\tau = t_2 - t_1$.

(a) Calculate the temporal Fourier transform $\mathcal{F}[G^{(1)}(\tau)]$ in terms of $E(\omega)$, the Fourier transform of the electric field. You should find that it is just the spectral density $S(\omega)$. (This is a hint – whatever you get in terms of the E's should agree with the spectral density.) Use this formula in the following parts.

(b) Calculate the correlation function $g^{(1)}(\tau)$ for a system with spectral density $S(\omega) = I_0\delta(\omega - \omega_0)$, a single-mode laser.

(c) Calculate the spectral density for a exponentially decaying coherence $g^{(1)}(\tau) = e^{-\Gamma\tau}$.

(d) Calculate the correlation function $g^{(1)}(\tau)$ for a Gaussian spectral density $S(\omega) = I_0 e^{-(\omega-\omega_0)^2/(2\Delta^2)}$.