Problem 1. Rabi spectroscopy, Fermi’s Golden Rule

This problem finishes off our understanding of the dynamics of driven two-level systems with periodic drives, and this completes essentially all we will need to know about driven, coherent two-level systems for a while.

In particular, consider a two-level atom with levels $|g\rangle$ and $|e\rangle$ with the Hamiltonian

$$H = \Delta |e\rangle\langle e| + \frac{1}{2} \Omega \cos(\omega t) |e\rangle\langle g| + \text{H.c.}$$

$$\approx \Delta |e\rangle\langle e| + \frac{\Omega}{2} e^{i\omega t} |g\rangle\langle e| + \text{H.c.},$$

assuming that $\omega$ is near $\Delta$ and making the rotating wave approximation in the second line. In the problem below, I will also use the spin language where $|\downarrow\rangle = |g\rangle$ and $|\uparrow\rangle = |e\rangle$.

In class we calculated the dynamics for the on-resonant case, and considered the dressed state resulting from adiabatic elimination in the far-off-resonant case; these correspond to $\delta \ll \Omega$ and $\delta \gg \Omega$ where $\delta \equiv \Delta - \omega$ is the detuning. Here you will develop and understand the behavior for general values of $\delta$ and $\Omega$.

As we shall see, this problem corresponds to considering the behavior of an atom when one shines a coherent light source (a laser) on it for a given time. This is a type of spectroscopy known as Rabi spectroscopy. Frequently, one fixes the time $t$ for which the laser is applied and varies $\omega$, calling this the spectrum.

Throughout this problem, you may for simplicity consider a system initial in the $|g\rangle$ state.

(a) Calculate the probability $P_e(t)$ of being in the state $|e\rangle$ as a function of time. At what frequency does the probability oscillate? This is called the “generalized Rabi frequency” or sometimes just the Rabi frequency. *Hint: it will be easier if you transform to new basis states to make $H$ time-independent as we did in class; solve the problem in that basis and transform back if necessary.*

(b) Consider the coherences measured by $\sigma^x$ and $\sigma^y$. At what frequency do these oscillate as a function of time? ($\delta$ and $\Omega$ are arbitrary.) *Hint: be careful regarding the coherences in the bare basis versus the basis to make $H$ time-independent.*

(c) For the far-off-resonant and the near-resonant cases, sketch the dynamics on the Bloch sphere of the spin formed by the $|g\rangle$ and $|e\rangle$ states in the bare atomic frame. Do the same for the rotating frame states $|\tilde{g}\rangle$ and $|\tilde{e}\rangle$ used to make $H$ time-independent.

(d) Draw the Rabi spectrum – $P_e$ as a function of $\delta$ – for a few different times $t$. You may consider $\Omega$ to be small if you’d like. Show enough times to characterize the different stages of evolution. Describe what sets the scales of the various features of the spectra; for example, these might be widths, heights, and areas of any peaks, wiggles, decays, etc.

(e) Consider the following (extremely useful) limit: $\Omega t \ll 1$ (i.e., “small pulse area”), so that $P_e(t)$ is small for all $\delta$, and $t$ larger. With this result, you should have derived Fermi’s Golden Rule: $\Gamma_{g\rightarrow e} = 2\pi |\langle e|H|g\rangle|^2 \delta (\omega - \Delta)$. *Hint: use the pictures of the
previous part and look for things that approach delta functions. As an aside, there are some subtle issues with the \( t \to \infty \) limit (which is why I didn’t specify \( t \) large compared to what), but basically Fermi’s Golden Rule should work whenever you integrate the rate \( \Gamma \) over a range of frequencies large compared to \( 1/t \).

Problem 2. Casimir effect

To see the effects of the quantum vacuum fluctuations, consider electromagnetism between two conducting plates at \( x = 0 \) and \( L \). Let’s work in a one dimensional universe for simplicity. We want to see how a third plate at \( a \ll L \) is attracted or repelled from the plate at \( x = 0 \). So we will assume the plates at \( x = 0 \) and \( L \) are fixed and calculate the energy as a function of the middle plate’s position, \( a \). We will work at \( T = 0 \) so that the electromagnetic field is in the vacuum state (i.e. no photons).

(a) Calculate the energy of the electromagnetic vacuum field in the region \((0, a)\). Do the same for the region \((a, L)\). Write these in terms a sum over electromagnetic modes. You can ignore polarization.

(b) You will find the sum for each of these terms diverges – how does it diverge?

(c) Nevertheless, the force should be finite – therefore, the divergences you’re finding cancel. Calculate the total energy and see how these divergences cancel. \textit{Hint: you may want to look up \( \zeta \) regularization to make things easier; or you can just power through the math to see how it works. Or ask me.}

(d) Given the electromagnetic energy \( U \), calculate the force on the plate at \( a \), given by \( F = -\partial_a U \).