Problem 1. Rabi spectroscopy, Fermi's Golden Rule

This problem finishes off our understanding of the dynamics of driven two-level systems with periodic drives, and this completes essentially all we will need to know about driven, coherent two-level systems for a while.

In particular, consider a two-level atom with levels $|g\rangle$ and $|e\rangle$ with the Hamiltonian

$$\begin{aligned} H &= \Delta\,|e\rangle\langle e| + \frac{1}{2}\Omega\cos(\omega t)\,|e\rangle\langle g| + \text{H.c.} \\ &\approx \Delta\,|e\rangle\langle e| + \frac{\Omega}{2}e^{i\omega t}\,|g\rangle\langle e| + \text{H.c.}, \end{aligned}$$

assuming that ω is near Δ and making the rotating wave approximation in the second line. In the problem below, I will also use the spin language where $|{\downarrow}\rangle = |g\rangle$ and $|{\uparrow}\rangle = |e\rangle$.

In class we calculated the dynamics for the on-resonant case, and considered the dressed state resulting from adiabatic elimination in the far-off-resonant case; these correspond to $\delta \ll \Omega$ and $\delta \gg \Omega$ where $\delta \equiv \Delta - \omega$ is the detuning. Here you will develop and understand the behavior for general values of δ and Ω.

As we shall see, this problem corresponds to considering the behavior of an atom when one shines a coherent light sources (a laser) on it for a given time. This is a type of spectroscopy known as *Rabi spectroscopy*. Frequently, one fixes the time t for which the laser is applied and varies ω, calling this the spectrum.

Throughout this problem, you may for simplicity consider a system initial in the $|g\rangle$ state.

(a) Calculate the probability $P_e(t)$ of being in the state $|e\rangle$ as a function of time. At what frequency does the probability oscillate? This is called the "generalized Rabi frequency" or sometimes just the Rabi frequency. *Hint: it will be easier if you transform to new basis states to make H time-independent as we did in class; solve the problem in that basis and transform back if necessary.*

(b) Consider the coherences measured by σ^x and σ^y. At what frequency do these oscillate as a function of time? (δ and Ω are arbitrary.) (You don't need to calculate their amplitude.) *Hint: be careful regarding the coherences in the bare basis versus the basis to make H time-independent.*

(c) For the far-off-resonant and the near-resonant cases, sketch the dynamics on the Bloch sphere of the spin formed by the $|g\rangle$ and $|e\rangle$ states in the bare atomic frame. Do the same for the rotating frame states $|\tilde{g}\rangle$ and $|\tilde{e}\rangle$ used to make H time-independent.

(d) Draw the Rabi spectrum – P_e as a function of δ – for a few different times t. You may consider Ω to be small if you'd like. Show enough times to characterize the different stages of evolution. Describe what sets the scales of the various features of the spectra; for example, these might be widths, heights, and areas of any peaks, wiggles, decays, etc.

(e) Consider the following (extremely useful) limit: $\Omega t \ll 1$ (i.e., "small pulse area"), so that $P_e(t)$ is small for all δ, and t larger. With this result, you should have derived Fermi's Golden Rule: $\Gamma_{g\to e} = 2\pi\,|\langle e|H|g\rangle|^2\,\delta(\omega - \Delta)$. *Hint: use the pictures of the* *previous part and look for things that approach delta functions.* As an aside, there are some subtle issues with the $t \to \infty$ limit (which is why I didn't specify t large compared to what), but basically Fermi's Golden Rule should work whenever you integrate the rate Γ over a range of frequencies large compared to $1/t$.

Problem 2. Casimir effect

To see the effects of the quantum vacuum fluctuations, consider electromagnetism between two conducting plates at $x = 0$ and L. Let's work in a one dimensional universe for simplicity. We want to see how a third plate at $a \ll L$ is attracted or repelled from the plate at $x = 0$. So we will assume the plates at $x = 0$ and L are fixed and calculate the energy as a function of the middle plate's position, a. We will work at $T = 0$ so that the electromagnetic field is in the vacuum state (i.e. no photons).

(a) Calculate the energy of the electromagnetic vacuum field in the region $(0, a)$. Do the same for the region (a, L). Write these in terms a sum over electromagnetic modes. You can ignore polarization.

(b) You will find the sum for each of these terms diverges – how does it diverge?

(c) Nevertheless, the force should be finite – therefore, the divergences you're finding cancel. Calculate the total energy and see how these divergences cancel. *Hint: you may want to look up ζ regularization to make things easier; or you can just power through the math to see how it works. Or ask me.*

(d) Given the electromagnetic energy U, calculate the force on the plate at a, given by $F = -\partial_a U$.