Locality of gapped ground states in systems with power-law decaying interactions

Zhiyuan Wang[1, 2] and Kaden R. A. Hazzard[1, 2, 3]

[1] *Department of Physics and Astronomy, Rice University, Houston, Texas 77005, USA*
[2] *Rice Center for Quantum Materials, Rice University, Houston, Texas 77005, USA*
[3] *Department of Physics and Astronomy, University of California, Davis, CA 95616, USA*

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It has been proved that in gapped ground states of locally-interacting quantum systems, the effect of local perturbations decays exponentially with distance. However, in systems with power-law ($1/r^\alpha$) decaying interactions, no analogous statement has been shown, and there are serious mathematical obstacles to proving it with existing methods. In this paper we prove that when α exceeds the spatial dimension D, the effect of local perturbations on local properties a distance r away is upper bounded by a power law $1/r^{\alpha_1}$ in gapped ground states, provided that the perturbations do not close the spectral gap. The power-law exponent α_1 is tight if $\alpha > 2D$ and interactions are two-body, where we have $\alpha_1 = \alpha$. The proof is enabled by a method that avoids the use of quasiadiabatic continuation and incorporates techniques of complex analysis. This method also improves bounds on ground state correlation decay, even in short-range interacting systems. Our work generalizes the fundamental notion that local perturbations have local effects to power-law interacting systems, with broad implications for numerical simulations and experiments.

I. INTRODUCTION AND OVERVIEW OF RESULTS

Locality is a fundamental principle that underlies many theories of nature. Loosely speaking, locality means that an object is influenced directly only by its immediate surroundings, and in particular, should be insensitive to actions taken far away. The precise quantitative statement of this principle takes different forms in different contexts. In quantum many-body dynamics, locality manifests itself in the form of a causality lightcone: roughly, if a local perturbation takes place at time $t = 0$, then at time t its effect must be within a ball region $r \leq vt$, where r is the distance and v is the maximal allowed speed of propagation of any physical particles or signals in the system. In relativistic quantum field theories, such a causality lightcone is guaranteed by Lorentz invariance, where v is the speed of light, and effects exactly vanish outside the lightcone. In non-relativistic quantum many-body systems with short-range interactions, the Lieb-Robinson bound (LRB) [1] guarantees an effective causality lightcone: the effect of local perturbations decays exponentially in $(r - vt)$, where the speed v depends on the microscopic details of the system [2–4].

Consequences of locality take a slightly different form for equilibrium properties of the quantum many-body system. An important case is on the effect of a local perturbation on ground states. Specifically, let $\hat{H}$ be the Hamiltonian and consider the effect of a local perturbation $\hat{V}_Y$ (supported on region Y) on a local observable $\hat{S}_X$, supported on a region X far from Y. Intuitively, we expect that the expectation value of $\langle \hat{S}_X \rangle$ measured in the perturbed ground state should not deviate significantly from its unperturbed value when the distance d_{XY} is large, i.e. the deviation

$$\delta\langle \hat{S}_X \rangle_{\hat{V}_Y} \equiv \langle \hat{S}_X \rangle_{\hat{H}+\hat{V}_Y} - \langle \hat{S}_X \rangle_{\hat{H}} \tag{1}$$

should be small in magnitude. The rigorous proof of this was pioneered by Hastings [5–7], who showed that for gapped ground states of a locally interacting Hamiltonian, $|\delta\langle \hat{S}_X \rangle_{\hat{V}_Y}|$ is upper bounded by a subexponential function in d_{XY}, provided that the perturbation does not close the spectral gap. The proof was based on the idea of quasiadiabatic continuation (QAC) [5–7], which relates the perturbed ground state $|G\rangle_{\hat{H}+\hat{V}_Y}$ to the unperturbed one by a quasilocal unitary evolution

$$|G\rangle_{\hat{H}+\hat{V}_Y} = \mathcal{T} e^{i\int_0^1 H_{\rm eff}(t)dt} |G\rangle_{\hat{H}}, \tag{2}$$

where the effective Hamiltonian $H_{\rm eff}(t)$ has subexponntial decaying interactions ($\mathcal{T}$ is the time-ordering operation). This immediately transforms the problem back to the dynamical case, where a generalized Lieb-Robinson bound implies that $|\delta\langle \hat{S}_X \rangle_{\hat{V}_Y}|$ decays subexponentially in d_{XY}. This bound is referred to as the LPPL principle (local perturbations perturb locally) [8] and has been strengthened to an exponential decay $|\delta\langle \hat{S}_X \rangle_{\hat{V}_Y}| \leq Ce^{-\mu_1 d_{XY}}$ [9][51], where C is a constant and μ_1 is given in Tab. I.

In recent years, there has been increasing interest in understanding the analogous consequences of locality from long-range, power-law ($1/r^\alpha$) decaying interactions, driven in part by the ubiquity of these interactions in many cold atom and molecule [10–14], Rydberg atom [15–21], and trapped ion [22–26] experiments, typically with $0 \leq \alpha \leq 6$, as well as the Coulomb interaction. The important question then arises: when long-range interactions are present, to what extent can we still expect locality in the senses described above to hold? The answer to this question is far from obvious, since long-range interactions can give rise to non-local behaviors of correlation functions for sufficiently small α [27, 28]. For the dynamical part, LRB has been successfully generalized to power-law interacting systems [29–34], implying generalized causality lightcones ($r \propto e^{vt}$ for $D < \alpha < 2D$ [2], $r = vt^\beta$ for $2D < \alpha < 2D + 1$ [29], and $r = vt$ for

$\alpha > 2D + 1$ [31, 32]).

However, the implications of locality for equilibrium systems are far less understood when power-law interactions are present, even in the important case of gapped ground states. This is partly due to the difficulties in generalizing Hastings' QAC to the power-law case: QAC has only been formulated for $\alpha > 2D + 2$ [35], an extremely restrictive condition and one rarely satisfied in the experimental systems of interest. Furthermore, even for $\alpha > 2D + 2$, the LPPL principle has never been proved, and it is expected that the above method with the QAC in Ref. [35] would lead to power law exponents in the resulting bounds that are not tight.

In this paper, we prove the LPPL principle for gapped ground states of lattice quantum systems where interactions are bounded by a power law $1/r^\alpha$ in distance r, with $\alpha > D$. To achieve this goal, we devise an alternative method that avoids the use of QAC Eq. (2) (thereby circumventing the aforementioned difficulty) and incorporates techniques of complex analysis. This method also improves the LPPL bounds for short-range interacting systems, and applies to degenerate (either exact or approximate) ground states as well. Our main result is roughly as follows: for perturbations $\hat{V}_Y$ that do not close the spectral gap,

$$|\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}| \leq \begin{cases} \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_1} \\ \mathbf{P}(d_{XY})e^{-\mu_1 d_{XY}}, \end{cases} \tag{3}$$

where $\langle\ldots\rangle$ is a uniform average over the (possibly degenerate) ground state subspace, the first line is for power-law systems and the second line is for short-range interacting systems, the exponents α_1, μ_1 are given in Tab. I, and throughout this paper we use $\mathbf{P}(x)$ to denote a polynomial in x with non-negative coefficients [but $\mathbf{P}(x)$ in different equations or in different parts of the same equation need not be the same]. We see that α_1 is equal to α if $\alpha > 2D$ and interactions are two-body, in which case our bound is qualitatively tight [up to the subleading prefactor $\mathbf{P}(\ln d_{XY})$] since it agrees with perturbation theory.

As one notable byproduct, the method we use to obtain these bounds also improves bounds on correlation decay [2, 36] of gapped (possibly degenerate) ground states: for arbitrary local operators $\hat{A}_X, \hat{B}_Y$, their connected correlation function is bounded by

$$|\langle\hat{A}_X\hat{B}_Y\rangle - \langle\hat{A}_X\rangle\langle\hat{B}_Y\rangle| \leq \begin{cases} \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_2} \\ \mathbf{P}(d_{XY})e^{-\mu_2 d_{XY}}, \end{cases} \tag{4}$$

where the exponents α_2, μ_2 are given in Tab. I. We see that our method improves earlier exponents, even in the case of short-range interacting systems, where our bound improves Ref. [2]'s bound by approximately a factor of 2 for $\Delta \ll v$.

Our results have profound implications on numerical simulations and experiments. For example, it has been pointed out [37] that the LPPL principle straightforwardly implies an upper bound on the finite size error of several numerical ground state algorithms, such as exact diagonalization [38, 39] and the density matrix renormalization group [40, 41]. Our results Eq. (3) imply that the finite size error of a local observable $\hat{S}$ in gapped ground state simulations decays in the linear dimension of the system L as

$$\delta\langle\hat{S}\rangle_L \equiv |\langle\hat{S}\rangle_L - \langle\hat{S}\rangle_\infty| \leq \begin{cases} \mathbf{P}(\ln L)/L^{\alpha_3} \\ \mathbf{P}(L)e^{-\mu_3 L}, \end{cases} \tag{5}$$

provided that the finite system is connected to the thermodynamic limit by a uniformly gapped path [37]. As in Eqs. (3,4) the first line is for power-law systems while the second line is for short-range interacting systems, and the constants α_3, μ_3 are given in Tab. I.

Our paper is organized as follows. Tab. I summarizes the exponents $\alpha_1, \alpha_2, \alpha_3, \mu_1, \mu_2, \mu_3$ in Eqs. (3,4,5) for various interaction ranges. In Sec. II we introduce our improved method, and use this method to bound the response of local observables in gapped non-degenerate ground states, and obtain the main result, Eq. (3). In Sec. III we generalize the bounds to gapped degenerate ground states. In Sec. IV we discuss the implications of our bounds in finite size numerical simulations and prove Eq. (5). In Sec. V we use our improved method to obtain tighter bounds on ground state correlation decay, Eq. (4). We conclude in Sec. VI.

II. LOCALITY OF PERTURBATIONS TO GAPPED NON-DEGENERATE GROUND STATES

Our set-up is as follows. Let Λ_L be an infinite sequence of D-dimensional finite lattices, labeled by the linear system size $L \in \mathbb{Z}$, with $N \propto L^D$ number of lattice sites in total. On each site $i \in \Lambda_L$ sits a quantum degree of freedom [can be fermionic, bosonic, or a quantum spin system] with local Hilbert space $\mathcal{H}_i$. The Hamiltonian H_L acts on the global Hilbert space $\mathcal{H}_L \equiv \bigotimes_{i\in\Lambda_L}\mathcal{H}_i$, and can be written in the generic form

$$H_L = \sum_{X\subset\Lambda_L} h_X, \tag{6}$$

where the summation is over all subsets of Λ_L and h_X is the local Hamiltonian supported on X [52] (we will later specify some locality condition on h_X which requires

Interaction	Prior bound		Our bound (LPPL and correlation decay have same exponents: $\alpha_1=\alpha_2,\ \mu_1=\mu_2$)	FSE bound
	LPPL	Correlation decay		
$1/r^\alpha$, $\alpha>D$	-	$\alpha_2=\frac{\alpha}{1+2v/\Delta}$ [2]	$\alpha_1=\frac{2\alpha}{\pi}\arcsin(\tanh\frac{\Delta\pi}{2v})$	if $\alpha>D+1$: $\alpha_3=\min(\alpha-D,\alpha_1+1-D)$ if $D<\alpha\le D+1$: $\alpha_3=\begin{cases}\alpha-D & \text{if } \alpha_1>D\\ \alpha_1+\alpha-2D & \text{if } \alpha_1\le D\end{cases}$
$1/r^\alpha,\alpha>2D$ two body	-	$\alpha_2=\alpha$ [33]	$\alpha_1=\alpha$	$\alpha_3=\alpha-D$
$e^{-\mu r}$	$\mu_1=\frac{\mu}{1+2\mu v/\Delta}$ [9]	$\mu_2=\frac{\mu}{1+2\mu v/\Delta}$ [2]	$\mu_1=\frac{2\mu}{\pi}\arcsin(\tanh\frac{\Delta\pi}{2\mu v})$	$\mu_3=\mu_1$

TABLE I: Summary of the constants α_1, μ_1 (LPPL bounds), α_2, μ_2 (correlation decay bounds), and α_3, μ_3 (finite size error bounds) for previous results compared with ours, for both power-law and short-range interacting systems. Our main result is the proof of the LPPL principle Eq. (3) for ground states of power-law interacting systems with spectral gap Δ, but we also significantly improved the bound for systems with exponentially-decaying interactions, as well as the constants α_2,μ_2 that appears in the correlation decay bounds Eq. (4). The FSE bound Eq. (5) with exponents α_3,μ_3 is a primary application of our main result [previously, there is only a FSE bound for short-range systems [37], in which $\mu_3=\mu_1=\mu/(1+2\mu v/\Delta)$]. v is a constant that appears in the LRB that can be straightforwardly calculated (for short-range interacting systems, v is the Lieb-Robinson speed).

$\|h_X\|$ to be small for large X). Throughout this section we assume that H_L has a non-degenerate ground state $|G_L\rangle$ with spectral gap Δ_L (the energy difference between the first excited state and the ground state) that is uniformly bounded from below, i.e. there exists $\Delta^{(0)}>0$ such that $\Delta_L\ge\Delta^{(0)}$ for all Λ_L. At this point we do not make assumptions on the range of interaction, nor do we assume that the local Hilbert space is finite dimensional.

Let V_Y be a local perturbation supported on region Y. Suppose that for all $\lambda\in[0,1]$, $H_L(\lambda)\equiv H_L+\lambda V_Y$ has a non-degenerate ground state $|G_L(\lambda)\rangle$ with spectral gap $\Delta_L(\lambda)$ that is uniformly bounded from below, i.e. $\exists\Delta>0$ such that $\forall\lambda\in[0,1]$, $\Delta_L(\lambda)\ge\Delta>0$, for all Λ_L. This condition will always be satisfied for sufficiently small perturbations satisfying $||V_Y||<\Delta^{(0)}/2$ ($||\cdots||$ is the operator norm), since Weyl's inequality [42] gives $\Delta_L(\lambda)\ge\Delta_L-2\lambda||V_Y||\ge\Delta^{(0)}-2||V_Y||$.

Let S_X be a local observable supported on region X such that $X\cap Y=\emptyset$. Our goal is to bound the response of S_X to the local perturbation V_Y, as defined in Eq. (1). We achieve this goal in two steps: in Sec. II A we present a general method to bound $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ using a Lieb-Robinson-type bound on the unequal time correlator $\langle G_L(\lambda)|[S_X(t),V_Y]|G_L(\lambda)\rangle$, where $S_X(t)=e^{iH_Lt}S_Xe^{-iH_Lt}$, and then in Secs. II C-II D we specialize to systems with different interaction ranges and apply the corresponding Lieb-Robinson bounds to obtain our main results in Eq. (3) and Tab. I. The resulting bounds are independent of the system size L, so they hold in the thermodynamic limit $L\to\infty$.

A. The improved method

In the following we present an improved method to bound $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ using a Lieb-Robinson-type bound on $\langle G_L(\lambda)|[S_X(t),V_Y]|G_L(\lambda)\rangle$. There are two main improvements compared to previous approaches: the first part generalizes the method in Ref. [37], which avoids the QAC and directly relates $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ to a specially constructed correlation function, while the second part obtains a bound on this correlation function from a LRB on $|\langle G_L(\lambda)|[S_X(t),V_Y]|G_L(\lambda)\rangle|$ using complex analysis techniques, which significantly improves the previous method in Ref. [37].

Since we have a gapped path for $\lambda\in[0,1]$, we can use perturbation theory to relate the rate of change of $\langle\hat{S}_X\rangle_{L,\lambda}\equiv\langle G_L(\lambda)|\hat{S}_X|G_L(\lambda)\rangle$ at each λ to a special correlation function, from which we will obtain an exact expression for $\delta\langle\hat{S}_X\rangle_{\hat{V}_Y}$ as an integral over the correlation function. Choose the normalization and phase of $|G_L(\lambda)\rangle$ such that $\langle G_L(\lambda)|G_L(\lambda)\rangle=1$ and $\langle G_L(\lambda)|\frac{d}{d\lambda}|G_L(\lambda)\rangle=0,\forall\lambda\in[0,1]$. For any finite L, first order non-degenerate perturbation theory gives the exact identity

$$\frac{d}{d\lambda}|G_L(\lambda)\rangle=\frac{\bar{P}_{G_L}(\lambda)}{\hat{H}_L(\lambda)-E_L(\lambda)}V_Y|G_L(\lambda)\rangle,\tag{7}$$

where $E_L(\lambda)$ is the ground state energy of $\hat{H}_L(\lambda)$ and $\bar{P}_{G_L}(\lambda)\equiv\hat{\mathbb{1}}-|G_L(\lambda)\rangle\langle G_L(\lambda)|$ is the projection operator to the space of excited states. Then

$$\frac{d}{d\lambda}\langle\hat{S}_X\rangle_{L,\lambda}=\langle G_L(\lambda)|\hat{S}_X\frac{\bar{P}_{G_L}(\lambda)}{\hat{\Delta}_L(\lambda)}\hat{V}_Y|G_L(\lambda)\rangle+\text{c.c.},\tag{8}$$

where $\hat{\Delta}_L(\lambda)\equiv\hat{H}_L(\lambda)-E_L(\lambda)$, whose spectrum is lower bounded by Δ. In the following we prove a uniform

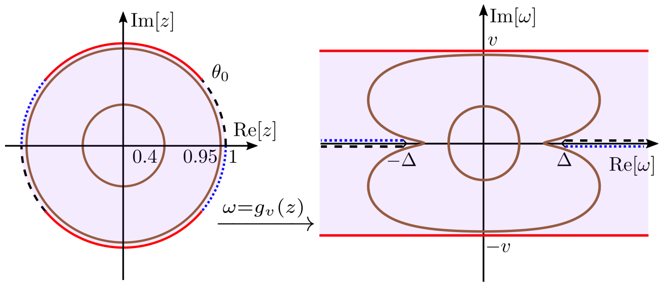

FIG. 1: For any finite system size L, K_Δ (dashed region on the real axis of the right panel) contains all possible pole positions of $\Omega_{XY}(\omega)$, so $\Omega_{XY}(\omega)$ is complex analytic in the region $\mathbb{C}\backslash K_\Delta$, as guaranteed by Eq. (9). The conformal mapping $\omega = g_v(z)$ [Eq. (18)] maps the unit disk (left) to the shaded region of the infinite strip with the pole regions excluded (right).

bound (independent of L, λ) on the RHS of Eq. (8), so that a bound on $\delta\langle \hat{S}_X\rangle_{\hat{V}_Y}$ immediately follows from $|\delta\langle \hat{S}_X\rangle_{\hat{V}_Y}| \le \int_0^1 d\lambda |d\langle \hat{S}_X\rangle_{L,\lambda}/d\lambda|$.

From now on we omit the labels L, λ. We define

$$\begin{aligned}\Omega_{XY}(\omega) \;&\equiv\; \int_0^{\eta_\omega\infty} \langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle e^{i\omega t}dt \qquad (9)\\ &=\; \langle G|\hat{S}_X \frac{i\bar{P}_G}{\omega-\hat{\Delta}}\hat{V}_Y|G\rangle - \langle G|\hat{V}_Y\frac{i\bar{P}_G}{\omega+\hat{\Delta}}\hat{S}_X|G\rangle,\end{aligned}$$

where $\eta_\omega = \mathrm{sgn}[\mathrm{Im}(\omega)]$. Notice that the RHS of Eq. (8) is exactly $i\Omega_{XY}(0)$. Taking the absolute value of the first line of Eq. (9) and using triangle inequality, we have

$$\begin{aligned}|\Omega_{XY}(\omega)| \;&\le\; \int_0^\infty |\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle| e^{-|\mathrm{Im}(\omega)|t}dt\\ &\le\; \int_0^\infty C(d_{XY},t)e^{-|\mathrm{Im}(\omega)|t}dt\\ &\equiv\; \bar{\Omega}(d_{XY},y), \qquad (10)\end{aligned}$$

where $y = |\mathrm{Im}[\omega]|$. In the second line of Eq. (10) we assume a Lieb-Robinson-type bound $|\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle| \le C(d_{XY},t)$, whose expression will be given in Secs. II C-II D when we consider systems with different range of interaction. At large t, $C(d_{XY},t)$ equals the constant trivial bound $2\|S_X\|\|V_Y\|$, so $\bar{\Omega}(d_{XY},y)$ is finite for any ω with $\mathrm{Im}(\omega) \ne 0$, but diverges as $\bar{\Omega}(d_{XY},y) \sim 1/y$ when $y \to 0$, so gives no bound on the desired $|\Omega_{XY}(0)|$.

Nevertheless, we can obtain a bound on $|\Omega_{XY}(0)|$ from the above by using a powerful technique from complex analysis. Notice that for any finite system size L, Eq. (9) guarantees that $\Omega_{XY}(\omega)$ is complex analytic in the region $\mathbb{C}\backslash K_\Delta$, where $K_\Delta = \{\omega \in \mathbb{R}|\omega \ge \Delta \text{ or } \omega \le -\Delta\}$, as shown in Fig. 1. The analytic structure allows us to improve the bound on $|\Omega_{XY}(\omega)|$ over the initial bound in Eq. (10), by applying the following lemma (Thm. 2.12 in Ref. [43]):

Lemma 1. *If $g(z)$ is complex analytic in a domain (a simply connected open region) S, then $u(z) = \ln|g(z)|$ is a subharmonic function in S, i.e. for any $z_0 \in S$ and $\rho > 0$, if the circular region defined by $|z - z_0| \le \rho$ is contained in S, then*

$$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\rho e^{i\theta})d\theta. \qquad (11)$$

To bound $|\Omega_{XY}(0)|$, let $f(\xi)$ be a complex analytic function in a domain S containing the unit disk D, such that $f(0) = 0$ and the image $f(D)$ is also a domain with $f(D) \cap K_\Delta = \emptyset$. Then $\Omega_{XY}[f(\xi)]$ is complex analytic for $\xi \in S$, so according to Lemma 1, $\ln|\Omega_{XY}[f(\xi)]|$ is subharmonic in S, therefore

$$\begin{aligned}\ln|\Omega_{XY}(0)| \;&\le\; \frac{1}{2\pi}\int_0^{2\pi}\ln|\Omega_{XY}[f(e^{i\theta})]|d\theta\\ &\le\; \frac{1}{2\pi}\int_0^{2\pi}\ln\bar{\Omega}[d_{XY},|\mathrm{Im}f(e^{i\theta})|]d\theta. \qquad (12)\end{aligned}$$

We will see that the integration over θ in the last line is convergent despite $\bar{\Omega}(d_{XY},y)$ diverging when $y \to 0$.

The rest of our task is to insert the LRB of specific systems into Eq. (10) to get $\bar{\Omega}(d_{XY},y)$, and then choose a suitable $f(\xi)$ to compute the second line of Eq. (12). Notice that the inequality (12) holds for all such functions $f(\xi)$ (satisfying the conditions mentioned above), so we will choose a $f(\xi)$ to optimize this bound.

B. Power-law interactions with $\alpha > 2D$

We start with the simplest case: $\alpha > 2D$ and all interactions are two body, i.e. all the h_X in Eq. (6) are of the form $h_X = h_{ij}V_iW_j$ where V_i, W_j are local operators with finite norm and finite support separated by a distance d_{ij} and h_{ij} are real parameters satisfying $h_{ij} \le \mathbf{C}d_{ij}^{-\alpha}$ [53]. Similar to the $\mathbf{P}(x)$ notation, throughout this paper we use $\mathbf{C}$ to denote a positive constant independent of r and t, and $\mathbf{C}$ in different equations or in different parts of the same equation need not be the same. In this case we use the Lieb-Robinson bound with an algebraic light cone [54] proved in Ref. [29]

$$C(r,t) \le \mathbf{C}\exp\left(vt - \frac{r}{t^\gamma+\mathbf{C}}\right) + \frac{\mathbf{C}t^{\alpha(1+\gamma)}+\mathbf{C}t^\alpha}{r^\alpha}, \qquad (13)$$

where $\gamma = (1+D)/(\alpha-2D)$. Inserting into Eq. (10) gives

$$\begin{aligned}\bar{\Omega}(r,y) \;&=\; \int_0^{t_0} C(r,t)e^{-yt}dt + \int_{t_0}^\infty \mathbf{C}e^{-yt}dt \qquad (14)\\ &\le\; \frac{\mathbf{C}t_0}{2}[e^{-r}+e^{(v-y)t_0-r/(t_0^\gamma+\mathbf{C})}] + \mathbf{C}\frac{e^{-yt_0}}{y}\\ &\quad+ \frac{1}{r^\alpha}\left[\mathbf{C}\frac{\Gamma(\alpha+1)}{y^{\alpha+1}} + \mathbf{C}\frac{\Gamma[\alpha(\gamma+1)+1]}{y^{\alpha(\gamma+1)+1}}\right],\end{aligned}$$

where for $t > t_0 \equiv (r/v)^{1/(\gamma+1)}$ we use the trivial bound $C(r,t) \le \mathbf{C}$, and in the second line we use Jensen's inequality since the integrand is convex. The second line in Eq. (14) decays subexponentially in r, while the last line decays algebraically, so the term proportional to $r^{-\alpha}$ dominates the long-distance behavior of $\bar{\Omega}(r,y)$. Inserting $\bar{\Omega}(r,y)$ into Eq. (12) with $f(\xi)=\xi$, one obtains Eq. (3) where the subleading factor $\mathbf{P}(\ln r)$ is a constant in this case. See App. A 1 for details.

C. Power-law interactions with $\alpha > D$

The bound in the previous section does not apply to the case $D<\alpha<2D$, and is limited to two-body (two-cluster) interactions. In this section we consider the more general case where the h_X in Eq. (6) satisfies [2]

$$\sum_{X:X\supset\{i,j\}} \|h_X\| \le \frac{h_0}{d_{ij}^{\alpha}}, \tag{15}$$

for all i and j, with $\alpha > D$. In this case, the Hastings-Koma bound [2] is the tightest general LRB:

$$\|[\hat{S}_X(t),\hat{V}_Y]\| \le \min\{\mathbf{C}\frac{e^{vt}}{d_{XY}^{\alpha}},\mathbf{C}\}, \tag{16}$$

where v is a positive constant. Using $|\langle G|[\hat{S}_X(t),\hat{V}_Y]|G\rangle| \le \|[\hat{S}_X(t),\hat{V}_Y]\|$, and substituting $C(d_{XY},t)$ in Eq. (10) by the RHS of Eq. (16) gives

$$\begin{aligned}\bar{\Omega}(r,y) &= \int_0^\infty C(r,t)e^{-yt}dt \\ &= \int_0^{t_0} \mathbf{C}\frac{e^{(v-y)t}}{r^\alpha}dt + \int_{t_0}^\infty \mathbf{C}e^{-yt}dt \\ &\le \mathbf{C}t_0\frac{1+e^{(v-y)t_0}}{2r^\alpha} + \mathbf{C}\frac{e^{-yt_0}}{y} \\ &= \frac{\mathbf{C}t_0}{2r^\alpha} + \mathbf{C}(\frac{t_0}{2}+\frac{1}{y})e^{-yt_0} \\ &\le \begin{cases}\mathbf{C}t_0e^{-yt_0}/y, & y\le v,\\ \mathbf{C}t_0r^{-\alpha}, & y>v.\end{cases}\end{aligned} \tag{17}$$

where we define $vt_0 = \ln\mathbf{C} + \alpha\ln r$ and the third line follows from Jensen's inequality since the integrand is convex.

Now we insert Eq. (17) into Eq. (12), with the conformal mapping $f(z)=g_v(bz)$ where $b\in(0,1)$ is a free parameter and [55]

$$g_v(z) = \frac{2v}{\pi}\,\mathrm{arctanh}\left(\frac{2z}{z^2+1}\tanh\frac{\Delta\pi}{2v}\right). \tag{18}$$

Eq. (12) becomes

$$\begin{aligned}\ln|\Omega_{XY}(0)| &\le \frac{2}{\pi}\int_0^{\pi/2}[\ln(\mathbf{C}t_0) - \ln y(\theta) - y(\theta)t_0]d\theta \\ &= \ln(\mathbf{C}t_0) - \frac{2}{\pi}\int_0^{\pi/2}[\ln y(\theta) + y(\theta)t_0]d\theta,\end{aligned} \tag{19}$$

where $y(\theta)=\mathrm{Im}[g_v(be^{i\theta})]$. The image of the unit disk under the mapping $\omega = g_v(z)$ is shown in Fig. 1. For $b<1$, the first term in the integral in Eq. (19) gives a constant independent of r, but this constant diverges as $b\to 1$. In the limit $b\to 1$, $y(\theta)$ becomes a step function: $y(\theta)=0$ for $\theta<\theta_0$ while $y(\theta)=v$ for $\theta>\theta_0$, where θ_0 satisfies $\cos\theta_0 = \tanh\frac{\Delta\pi}{2v}$ and is marked in Fig. 1. Then in the limit $b\to 1$ the second term in the integral in Eq. (19) is

$$\begin{aligned}-t_0\frac{2}{\pi}\int_0^{\pi/2} y d\theta &= -2t_0v(\pi/2-\theta_0)/\pi \\ &= -\frac{2t_0v}{\pi}\arcsin\left(\tanh\frac{\Delta\pi}{2v}\right).\end{aligned} \tag{20}$$

Putting things together, we get

$$|\Omega_{XY}(0)| \le P_b(\ln r)r^{-\alpha_1(b)}, \tag{21}$$

where $P_b(\ln r)$ is a polynomial in $\ln r$ for any $b<1$, but its coefficients diverge as $b\to 1$, while

$$\lim_{b\to 1}\alpha_1(b) = \frac{2\alpha}{\pi}\arcsin\left(\tanh\frac{\Delta\pi}{2v}\right).$$

Minimizing the RHS of Eq. (21) with respect to b for each r, we obtain the result in Eq. (3) and Tab. I. [The $\mathbf{P}(\ln r)$ factor may have a higher degree than $P_b(\ln r)$.] See App. A 2 for a detailed derivation.

D. Short-range interacting systems

The method in Sec. II A also significantly improves the LPPL bounds for systems with short range interactions, either exponentially decaying or strictly finite ranged. Specifically, we consider systems whose Hamiltonians Eq. (6) satisfy [2]

$$\sum_{X:X\supset\{i,j\}} \|h_X\| \le h_0e^{-\mu d_{ij}}, \tag{22}$$

for all i and j, where μ is some positive constant. The Lieb-Robinson bound is [2]

$$C(r,t) \le \mathbf{C}e^{-\mu(r-vt)}. \tag{23}$$

Notice that the RHS of Eq. (23) can be obtained from the RHS of Eq. (16) with the substitutions $r\to e^r, \alpha\to\mu, v\to\mu v$. We can therefore directly make this substitution in the results of Sec. II C and obtain the bound

$$|\Omega_{XY}(0)| \le \mathbf{P}(r)e^{-\mu_1 r} \tag{24}$$

with μ_1 given in Tab. I. We see that for $\Delta \ll v$ our bound gives $\mu_1\approx\Delta/v$, which improves the previous best bound $\mu_1 = \mu/(1+2\mu v/\Delta)\approx\Delta/(2v)$ by approximately a factor of 2. Furthermore, if one wants a tighter bound for a specific model, one can use the LRB in Eq. (32) of

Ref. [4]: $C(r,t) \le \mathbf{C} e^{\omega_m(i\kappa)t-\kappa r}, \forall \kappa > 0$, where $\omega_m(i\kappa)$ is some (efficiently computable) function of κ (Ref. [4] mainly deals with systems with finite range interactions, but the method can be directly generalized to systems with exponentially decaying interactions). This leads to a bound of the same form as Eq. (24) in which μ_1 is a function of κ. One can then maximize $\mu_1(\kappa)$ over $\kappa > 0$. This method gives further quantitative improvement for a specific model, especially at large Δ/v.

III. GENERALIZATION TO GAPPED DEGENERATE GROUND STATES

In this section we generalize our bounds to gapped systems with degenerate ground states. We begin with a straightforward extension. Notice that if the system has a subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ such that both the Hamiltonian H and the perturbation V_Y leave $\mathcal{H}_1$ invariant (this is not required for S_X), and the ground state $|G_1\rangle$ of $\mathcal{H}_1$ is non-degenerate and gapped (within $\mathcal{H}_1$), then all our proofs in the previous section applies to this subspace $\mathcal{H}_1$, provided that $\bar{P}_G$ in Eq. (7) is understood as the projector to all the excited states within $\mathcal{H}_1$. In particular, if the system has a set of conserved quantum numbers that commute with both H and V_Y and distinguish all the gapped degenerate ground states, then our bounds apply to all the ground states.

Nevertheless, this simple extension does not apply if the perturbation V_Y breaks the conserved quantities. It also fails if the degeneracy is not due to any symmetry at all, which includes the important class of topological degeneracy, where the (approximately) degenerate ground states cannot be distinguished by local conserved quantum numbers. In the following we present a more general treatment for degenerate ground states (motivated by the method in Ref. [44]), which shows that all our results in Tab. I still hold provided that $\langle S_X \rangle$ is averaged over all the (nearly) degenerate ground states with equal weights. This can be thought of as the temperature $T \to 0$ limit of the statistical mechanical average, as long as this limit is taken after the thermodynamic limit $L \to \infty$, in which the splitting of ground state degeneracy vanishes.

Let us denote the degenerate ground states of $H(\lambda) = H + \lambda V_Y$ as $|G^a(\lambda)\rangle$, with energy $E_0^a(\lambda)$, for $a = 1, 2, \ldots, d$, respectively. Notice that we do not require the degeneracy to be exact (which is important for treating topological degeneracy), but only that at each λ, all the ground state energies $E_0^a(\lambda)$ are separated from the rest of the spectrum (the excited states) by at least an amount $\Delta(\lambda) > 0$, and $\Delta(\lambda)$ is uniformly bounded from below, i.e. $\Delta \equiv \inf_{\lambda\in[0,1]} \Delta(\lambda) > 0$. [Similar to the non-degenerate case, as long as $\Delta(0) > 0$, the uniform gap condition is always satisfied for sufficiently small $\|V_Y\|$, as guaranteed by Weyl's inequality.]

The method follows Sec. II A, but now using degenerate perturbation theory. If some of the ground states are exactly degenerate at some λ, then we have some freedom to choose a basis for the exactly degenerate subspace, and it can be shown that [44] it is always possible to choose a suitable basis for the this subspace such that V_Y is diagonal within this subspace and $\langle G^a(\lambda)|\partial_\lambda|G^b(\lambda)\rangle = 0$ whenever $E_0^a(\lambda) = E_0^b(\lambda)$. Then degenerate perturbation theory generalizes Eq. (7) to

$$\partial_\lambda |G^a(\lambda)\rangle = \frac{\bar{P}^a(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)} V_Y |G^a(\lambda)\rangle, \tag{25}$$

where

$$\begin{aligned} \bar{P}^a(\lambda) &= \mathbb{1} - \sum_{b: E_0^b(\lambda) = E_0^a(\lambda)} |G^b(\lambda)\rangle\langle G^b(\lambda)| \\ &= \bar{P}_G(\lambda) + \sum_{b: E_0^b(\lambda) \neq E_0^a(\lambda)} |G^b(\lambda)\rangle\langle G^b(\lambda)|, \end{aligned} \tag{26}$$

where $\bar{P}_G(\lambda) \equiv \hat{\mathbb{1}} - \sum_{b=1}^d |G^b(\lambda)\rangle\langle G^b(\lambda)|$ is the projection operator to the space of all excited states. Inserting the second line of Eq. (26) into Eq. (25), we get

$$\partial_\lambda |G^a(\lambda)\rangle = \frac{\bar{P}_G(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)} V_Y |G^a(\lambda)\rangle + \sum_{b=1}^d Q^{ab} |G^b(\lambda)\rangle, \tag{27}$$

where

$$Q^{ab} = \begin{cases} \frac{\langle G^b(\lambda)|V_Y|G^a(\lambda)\rangle}{E_0^b(\lambda) - E_0^a(\lambda)}, & \text{if } E_0^b(\lambda) \neq E_0^a(\lambda), \\ 0, & \text{if } E_0^b(\lambda) = E_0^a(\lambda), \end{cases} \tag{28}$$

is an anti-Hermitian matrix $(Q^{ab})^* = -Q^{ba}$. We now consider the expectation value $\langle S_X \rangle_\lambda$ of a local observable S_X averaged over all degenerate ground states $\{|G^b(\lambda)\rangle\}_{b=1}^d$, i.e. we define $\langle O \rangle_\lambda \equiv \frac{1}{d}\sum_{b=1}^d \langle G^b(\lambda)|O|G^b(\lambda)\rangle$ for any operator O. Then Eq. (8) becomes

$$\partial_\lambda \langle \hat{S}_X \rangle_\lambda = \left\langle \hat{S}_X \frac{\bar{P}_G(\lambda)}{\hat{H}(\lambda) - E_0^a(\lambda)} \hat{V}_Y \right\rangle_\lambda + \text{c.c.}, \tag{29}$$

where, importantly, the contribution of the second term in Eq. (27) cancel due to anti-Hermiticity of Q^{ab}. The rest of Sec. II A generalizes in a straightforward way, with the only difference being that the ground state expectation value $\langle G(\lambda)| \ldots |G(\lambda)\rangle$ is replaced by the average $\langle \ldots \rangle_\lambda$. Lieb-Robinson bounds can still be used as we have $\langle [\hat{S}_X(t), \hat{V}_Y] \rangle_\lambda \le \|[\hat{S}_X(t), \hat{V}_Y]\| \le C(r,t)$. All resulting bounds remain the same as those listed in Tab. I.

IV. IMPLICATIONS FOR FINITE SIZE NUMERICAL SIMULATIONS

In this section we present a straightforward application of our results, bounding the finite size errors (FSEs) of local observables in gapped ground states of power-law systems, generalizing the bounds for locally-interacting systems proved in Ref. [37]. The basic configuration for the 1D case is illustrated in Fig. 2. The FSE for a local observable $\hat{S}_X$ measured in a L-site calculation is defined as $\delta\langle\hat{S}_X\rangle_L \equiv |\langle\hat{S}_X\rangle_L - \langle\hat{S}_X\rangle_\infty|$, which can be considered as the effect of the boundary interaction $\hat{V}_Y$ on $\hat{S}_X$, since removing $\hat{V}_Y$ from the thermodynamic Hamiltonian $\hat{H}$ decouples the finite system and the outside, leading to $\langle\hat{S}_X\rangle_L = \langle\hat{S}_X\rangle_{\hat{H}-\hat{V}_Y}$. We assume that the spectral gap $\Delta_L(\lambda)$ of the interpolated Hamiltonian $\hat{H}-\lambda\hat{V}_Y$ is uniformly bounded from below $\min_{\lambda\in[0,1]}\Delta_L(\lambda) = \Delta > 0$ [56]. Under this assumption, we can apply our main result Eq. (3) to upper bound $\delta\langle\hat{S}_X\rangle_L$. A complication here is that $\hat{V}_Y$ contains infinitely many terms, including those that are very close to $\hat{S}_X$, so $r = d_{XY}$ is zero. To solve this issue, we can write

$$\hat{V}_Y = \sum_{i\in L, j\notin L} \hat{V}_{ij}, \tag{30}$$

where the summation is over all the interaction terms $\hat{V}_{ij}$ with i in the L-site system and j outside. Inserting Eq. (30) into Eq. (8), and using Eqs. (9-12) to upper bound the contribution of each individual $\hat{V}_{ij}$ term independently, we get

$$|\delta\langle\hat{S}_X\rangle_L| \le \sum_{i\in L, j\notin L} \|\hat{V}_{ij}\|\mathbf{P}(\ln r_{iX})/r_{iX}^{\alpha_1}. \tag{31}$$

In the following we treat the 1D case for simplicity, and present the derivation in arbitrary dimension in App. B 2. Let $R = L/2$ and $\delta(r) = \mathbf{P}(\ln r)/r^{\alpha_1}$, we have

$$\begin{aligned}|\delta\langle\hat{S}_X\rangle_L| &\le \sum_{-R\le i\le R, |j|>R} \delta(|i|+1)/(j-i)^{\alpha} \\ &\le \sum_{-R\le i\le R} \mathbf{C}\delta(|i|+1)/(R+1-i)^{\alpha-1} \\ &\le \sum_{i=1}^{R+1} \mathbf{P}(\ln i) i^{-\alpha_1}(R+2-i)^{1-\alpha}. \end{aligned} \tag{32}$$

The following lemma gives a bound for the convolutional sum (see App. B 1 for proof):

Lemma 2. *Let η, ζ be real constants satisfying $0 < \eta \le \zeta$. Then*

$$\sum_{r=1}^{R-1} \frac{\mathbf{P}(\ln r)}{r^\zeta (R-r)^\eta} \asymp \mathbf{P}(\ln R) \times \begin{cases} R^{-\eta}, & \text{if } \zeta \ge 1, \\ R^{1-\eta-\zeta}, & \text{if } \zeta < 1, \end{cases} \tag{33}$$

where the notation $f(R) \asymp g(R)$ means that there exist positive constants c_1, c_2 independent of R such that $c_1 g(R) \le f(R) \le c_2 g(R)$ for all $R \in \mathbb{Z}_{\ge 1}$.

Applying Lemma 2 to Eq. (32), we obtain Eq. (5) with

$$\alpha_3 = \begin{cases} \alpha_1 + \alpha - 2 & \text{if } \alpha_1 \le 1 \\ \alpha - 1 & \text{if } \alpha_1 > 1 \end{cases}$$

for $1 < \alpha \le 2$, and $\alpha_3 = \alpha - 1$ for $\alpha > 2$, which is the result in Tab. I for $D = 1$.

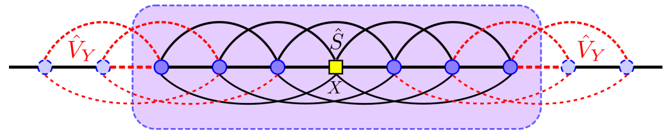

FIG. 2: Upper bounding finite-size error with the LPPL, illustrated for a 1D chain. The LPPL principle immediately gives an upper bound on finite size error of local observables in numerical simulation of gapped ground states, by recognizing $\hat{V}_Y$ as the interactions between the sites of the finite system and sites lying outside.

V. IMPROVED BOUNDS ON GROUND STATE CORRELATION DECAY

In this section we show that the method in Sec. II A also significantly improves bounds on correlation decay of gapped (possibly degenerate) ground states, compared to previous results [2, 36]. We first obtain an integral formula that relates $\Omega_{XY}(\omega)$ in Eq. (9) and the connected correlation function $\langle S_X V_Y\rangle_c \equiv \langle S_X V_Y\rangle - \langle S_X\rangle\langle V_Y\rangle$ in the gapped ground state $|G\rangle$. Integrating the second line of Eq. (9) along the imaginary axis, we have

$$\begin{aligned}\int_{-\infty i}^{+\infty i} \Omega_{XY}(\omega) d\omega &= \int_{-\infty i}^{+\infty i} d\omega \langle G|\hat{S}_X \frac{i\bar{P}_G}{\omega - \hat{\Delta}} \hat{V}_Y|G\rangle \\ &\quad - \int_{-\infty i}^{+\infty i} d\omega \langle G|\hat{V}_Y \frac{i\bar{P}_G}{\omega + \hat{\Delta}} \hat{S}_X|G\rangle \\ &= \pi\langle G|\hat{S}_X\bar{P}_G\hat{V}_Y|G\rangle + \text{c.c.} \\ &= 2\pi\langle S_X V_Y\rangle_c, \end{aligned} \tag{34}$$

where we used the following equality

$$\int_{-\infty i}^{+\infty i} \frac{1}{\omega - \mu} d\omega = -\pi i \ \text{sgn}(\mu). \tag{35}$$

With Eq. (34), we can obtain an upper bound on $|\langle S_X V_Y\rangle_c|$ by integrating $|\Omega_{XY}(\omega)|$ along the imaginary axis. Furthermore, it can be proved that (see App. C) $|\Omega_{XY}(\omega)|$ on the imaginary axis can always be upper bounded by the upper bound of $|\Omega_{XY}(0)|$ obtained by Eq. (12) (we denote this upper bound by $|\bar{\Omega}_{XY}(0)|$). Therefore, we can use the upper bound $|\Omega_{XY}(iy)| \le \min[|\bar{\Omega}_{XY}(0)|, \bar{\Omega}(d_{XY}, y)]$. Notice that the integration on this bound on iy is guaranteed to converge provided one uses the best LRB, since $C(d_{XY}, t) \propto t^\nu$ at small t with $\nu \ge 1$, and so $\bar{\Omega}(d_{XY}, y)$ in Eq. (10) decays at least as $y^{-\nu-1}$ at large y. This upper bound yields

$$\begin{aligned} 2\pi|\langle S_X V_Y\rangle_c| &\le \int_{-\infty}^{+\infty} |\Omega_{XY}(iy)| dy \\ &\le 2y_0|\bar{\Omega}_{XY}(0)| + 2\int_{y_0}^{\infty} \bar{\Omega}(d_{XY}, y) dy, \end{aligned} \tag{36}$$

for any $y_0 > 0$ [for the optimal result, y_0 should satisfy $\bar{\Omega}(d_{XY}, y_0) = |\bar{\Omega}_{XY}(0)|$].

For example, for $D < \alpha < 2D$, we have

$$\bar{\Omega}(r,y) \le \frac{\mathbf{C}}{r^\alpha}\left[\frac{e^{(v-y)t_0}-1}{v-y} + \frac{e^{-yt_0}-1}{y}\right] + \mathbf{C}\frac{e^{-yt_0}}{y}. \tag{37}$$

Inserting Eq. (37) into Eq. (36) and taking $y_0 = v$, we see that the integral of the term in the square bracket converges to a constant independent of r, and therefore the second term in the last line of Eq. (36) is bounded by $\mathbf{C}/d_{XY}^\alpha$. For $|\bar{\Omega}_{XY}(0)|$ we use the result of Sec. II C [Eq. (21) with the optimal b]. In the end we obtain

$$|\langle S_X V_Y\rangle_c| \le \mathbf{P}(\ln d_{XY})/d_{XY}^{\alpha_1}. \tag{38}$$

Other cases in Tab. I can be treated in an identical manner, by inserting the results of Sec. II into Eq. (36). In all cases, one obtains Eq. (4) with $\alpha_2 = \alpha_1$ for the power-law cases or $\mu_2 = \mu_1$ for short-range interacting cases.

VI. CONCLUSION

We have proved a locality principle for gapped ground states in systems with power-law ($1/r^\alpha$) decaying interactions: when $\alpha > D$, the response of a local observable S_X to a spatially separated local perturbation V_Y decays as a power-law ($1/r^{\alpha_1}$) in distance, provided that V_Y does not close the spectral gap. When $\alpha > 2D$, the bound on the exponent α_1 that we obtain, $\alpha_1 = \alpha$, is tight. We proved this using a method that avoids the use of QAC and incorporates techniques of complex analysis. Our method also improves bounds on ground state correlation decay, even in short-range interacting systems.

Our results have profound significance in studying the ground state properties of power-law interacting systems. At a fundamental level, the LPPL bounds generalize the notion of locality to gapped ground states of power-law systems, implying that the local properties of such ground states are stable against distant local perturbations. At a more practical level, we showed how our results immediately lead to an upper bound on finite size error in numerical simulations of gapped ground states, which revealed that FSEs generally decay as a power-law ($1/L^{\alpha_3}$) in system size (provided that α or the spectral gap Δ is not too small). A corollary of this is the existence of thermodynamic limit for local observables in ground states of power-law systems, under the spectral gap assumption stated in Sec. IV.

We now discuss some open questions and future directions. One open question concerns whether the power law exponents α_1 and α_2 given in Tab. I are tight when $D < \alpha < 2D$: we see that in this case both of them are strictly smaller than α, yet for all gapped power-law systems we know, no correlations decay slower than $1/r^\alpha$, which strongly suggests that our bounds can further be improved in this case. An interesting future direction is to generalize our results to systems of interacting bosons, such the Bose-Hubbard model, where our current bounds do not apply due to the interaction h_X in Eq. (6) having infinite norm, thereby violating Eq. (15) and the corresponding LRBs. However, our method in Sec. II A may still work if we incorporate Eq. (10) with recent LR-type bounds for interacting bosons [3, 45–47]. It will then be interesting to see how the exponents in Tab. I get modified. Another future direction is to prove the stability of the spectral gap against extensive local perturbations, for example a small external field on every site in TFIM. For locally interacting systems, this has been proved for gapped frustration-free ground states under the local topological quantum order condition [48–50], where an essential tool in the proof is Hastings' QAC Eq. (2). It is interesting to investigate if our new method can improve these results and extend them to power-law systems.

Appendix A: Some details for Sec. II

In this appendix we provide some technical details for Sec. II.

1. From Eq. (14) to Eq. (3)

Our task here is to insert Eq. (14) into Eq. (12) to prove Eq. (3). We first simplify the last line of Eq. (14): notice that for $y = \mathrm{Im} f(e^{i\theta}) = \sin\theta < 1$, we have $e^{(v-y)t_0 - r/(t_0^\gamma + \mathbf{C})} \le \mathbf{C}e^{-yt_0}/y$, $r^{-\alpha}y^{-\alpha-1} \le \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}$, and $t_0 e^{-r} \le \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}$. Therefore

$$\bar{\Omega}(r,y) \le (\mathbf{C}t_0 + \mathbf{C}y^{-1})e^{-yt_0} + \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}. \tag{A1}$$

The second term in Eq. (A1) dominates at small and large y, while the first term is only important in an intermediate region (y_1, y_2), where $y_{1,2} = x_{1,2}r^{-1/(\gamma+1)}$ and x_1, x_2 are the two solutions to the equation (and are independent of r)

$$(x + \mathbf{C})e^{-xv^{-1/(\gamma+1)}} = x^{-\alpha(\gamma+1)}. \tag{A2}$$

In summary,

$$\bar{\Omega}(r,y) \le \begin{cases} (\mathbf{C}t_0 + \mathbf{C}y^{-1})e^{-yt_0}, & y_1 \le y \le y_2 \\ \mathbf{C}r^{-\alpha}y^{-\alpha(\gamma+1)-1}, & 0 < y < y_1 \text{ or } y > y_2. \end{cases} \tag{A3}$$

[In case Eq. (A2) has no solution, then $\bar{\Omega}(r,y)$ is always bounded by the second line of Eq. (A3), and our following derivations still work with minor modifications.] Inserting Eq. (A3) into Eq. (12), we have

$$\begin{aligned} \ln|\Omega_{XY}(0)| \le\ & \ln\mathbf{C} - \frac{2\alpha}{\pi}(\pi/2 - \theta_2 + \theta_1)\ln r \\ & + \int_{\theta_1}^{\theta_2}\left[\ln\left(\mathbf{C}t_0 + \frac{\mathbf{C}}{\sin\theta}\right) - t_0\sin\theta\right]d\theta, \end{aligned} \tag{A4}$$

where $y_{1,2} \equiv \sin\theta_{1,2} = x_{1,2}r^{-1/(\gamma+1)}$. Using $\theta_{1,2} = O[r^{-1/(\gamma+1)}]$, we see that all but the $\ln \mathbf{C} - \alpha \ln r$ term are of order $r^{-1/(\gamma+1)}$, $r^{-1/(\gamma+1)}\ln r$, or $r^{-2/(\gamma+1)}$, all of which are upper bounded by a constant for $r \geq 1$. This proves Eq. (3) with the subleading factor $\mathbf{P}(\ln r)$ being a constant.

2. From Eq. (19) to Eq. (3)

In Sec. II C we analyzed the asymptotic behavior of Eq. (19) at large r and argued that it leads to Eq. (3) with α_1 given in Tab. I. In this section we provide a more detailed and rigorous derivation, which in addition gives the polynomial prefactor (up to an overall multiplicative constant).

We begin by upper bounding the first term in the integrand in Eq. (19). To this end, we obtain a simple lower bound for $y(\theta)$ as follows:

$$\begin{aligned} y(\theta) &= \frac{2v}{\pi}\mathrm{Im}\left[\mathrm{arctanh}\left(\frac{2z}{z^2+1}\tanh\frac{\Delta\pi}{2v}\right)\right] \\ &\geq \frac{2v}{\pi}\arctan\left[\mathrm{Im}\left(\frac{2z}{z^2+1}\right)\tanh\frac{\Delta\pi}{2v}\right] \\ &= \mathbf{C}\arctan\left[\mathbf{C}\frac{(1-b^2)b\sin\theta}{b^4+2b^2\cos 2\theta+1}\right] \\ &\geq \mathbf{C}(1-b)\sin\theta, \end{aligned} \tag{A5}$$

for $b \geq 0.9$, where in the second line we used $\mathrm{Im}[\mathrm{arctanh}\, x] \geq \arctan \mathrm{Im}[x]$ (which follows from the fact that $\mathrm{Im}[\mathrm{arctanh}(x+i\epsilon)]$ is monotonically increasing in x for $\epsilon > 0, x > 0$), and the proof for the last line is elementary. Therefore the first term in the integrand in Eq. (19) can be upper bounded by

$$-\frac{2}{\pi}\int_0^{\pi/2}\ln y(\theta)d\theta \leq \ln\mathbf{C} - \ln(1-b). \tag{A6}$$

This may be a crude bound, but it captures the leading singularity of this term as $b \to 1$. We now study the second term in the integrand in Eq. (19) near $b \to 1$. We have

$$\begin{aligned} \partial_b y(\theta) &= \mathrm{Im}[\partial_b g_v(z)] \\ &= \mathrm{Im}\left[\frac{1}{ib}\partial_\theta g_v(z)\right] \\ &= -\frac{1}{b}\mathrm{Re}[\partial_\theta g_v(z)], \end{aligned} \tag{A7}$$

therefore

$$\begin{aligned} \partial_b \int_0^{\pi/2} y(\theta)d\theta &= -\frac{1}{b}\int_0^{\pi/2} d\theta\partial_\theta \mathrm{Re}[g_v(z)] \\ &= -\frac{1}{b}\mathrm{Re}[g_v(ib) - g_v(b)] \\ &= \frac{1}{b}g_v(b), \end{aligned} \tag{A8}$$

the limit of which at $b \to 1$ is Δ. Since this derivative exists for $b \in [b_0, 1]$ for any $b_0 > 0$, along with Eq. (20), we obtain

$$\int_0^{\pi/2} yd\theta \geq C_0 - \mathbf{C}(1-b), \tag{A9}$$

where $C_0 \equiv v\arcsin[\tanh(\Delta\pi/2v)]$. Inserting Eqs. (A6,A9) into Eq. (19), we get

$$\ln|\Omega_{XY}(0)| \leq -\alpha_1 \ln r + \ln\frac{\mathbf{C}t_0}{1-b} + \mathbf{C}t_0(1-b). \tag{A10}$$

Minimizing the RHS of Eq. (A10) [the minimum is at $1-b = 1/(\mathbf{C}t_0)$], we get Eq. (3) where the polynomial prefactor can be taken as $P(\ln r) = \mathbf{C}(\ln r)^2$.

Appendix B: Finite-size error bounds

In this appendix section we provide some missing details in Sec. IV, including a proof of Lemma 2 and a derivation of the bounds in arbitrary spatial dimension.

1. Proof of Lemma 2

For simplicity we assume $R = 2R_1 + 1$ is an odd number (the proof for even R is similar). We have

$$\begin{aligned} \sum_{r=1}^{R-1}\frac{\mathbf{P}(\ln r)}{r^\beta(R-r)^\alpha} &= \left(\sum_{r=1}^{R_1} + \sum_{r=R_1+1}^{R-1}\right)\frac{\mathbf{P}(\ln r)}{r^\beta(R-r)^\alpha} \\ &= \sum_{r=1}^{R_1}\left\{\frac{\mathbf{P}(\ln r)}{r^\beta(R-r)^\alpha} + \frac{\mathbf{P}[\ln(R-r)]}{r^\alpha(R-r)^\beta}\right\} \\ &\asymp \sum_{r=1}^{R_1}\left[\frac{\mathbf{P}(\ln r)}{r^\beta R^\alpha} + \frac{\mathbf{P}(\ln R)}{r^\alpha R^\beta}\right], \\ &\leq \sum_{r=1}^{R_1}\left[\frac{\mathbf{P}(\ln R)}{r^\beta R^\alpha} + \frac{\mathbf{P}(\ln R)}{r^\alpha R^\beta}\right], \end{aligned} \tag{B1}$$

where in the second line we substituted r by $R-r$ in the second sum, and in the third line we used $\mathbf{P}[\ln(R-r)](R-r)^{-\gamma} \asymp \mathbf{P}(\ln R)R^{-\gamma}$ for $1 \leq r \leq R_1$ and $\gamma > 0$ since $\mathbf{P}[\ln(R/2)] \leq \mathbf{P}[\ln(R-r)] \leq \mathbf{P}(\ln R)$ and $R^{-\gamma} \leq (R-r)^{-\gamma} \leq (R/2)^{-\gamma}$. Now applying $\sum_{r=1}^{R_1} r^{-\gamma} \asymp \int_{r=1}^{R_1} r^{-\gamma}dr$ to the last line of Eq. (B1) and calculating the integral, we obtain Eq. (33).

2. Derivation of the bounds in higher dimension

In Sec. IV we derived the finite size error bound Eq. (5) in 1D. In the following we generalize the derivation to arbitrary spatial dimension. The configuration is shown in Fig. 3. Without loss of generality we can assume that the system has a spherical shape (the sphere S in Fig. 3),

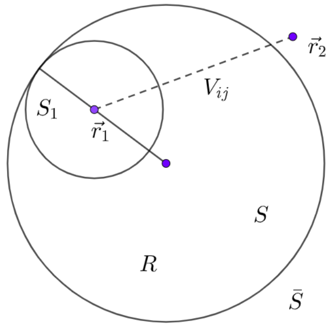

FIG. 3: Derivation of the finite-size error bound in higher spatial dimension. S is the finite system with radius R, $\overline{S}$ is its complement, $\vec{r}_1, \vec{r}_2$ are respectively the positions i, j of the power-law interaction V_{ij}, $S_1 \subseteq S$ is a subsystem centered at $\vec{r}_1$ with radius $R - r_1$ (so that S_1 touches S).

since the error in other cluster shapes can be upper and lower bounded by spheres with radius proportional to its linear dimension. Eq. (31) is still valid, so we have

$$\begin{aligned} |\delta\langle \hat{S}_X\rangle_L| &\le \sum_{\vec{r}_1\in S, \vec{r}_2\in\overline{S}} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}|\vec{r}_2-\vec{r}_1|^{\alpha}} \\ &\le \sum_{\vec{r}_1\in S, \vec{r}_2\in\overline{S_1}} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}|\vec{r}_2-\vec{r}_1|^{\alpha}} \\ &\le \sum_{\vec{r}_1\in S} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1}(R-r_1)^{\alpha-D}} \\ &\le \sum_{r_1=1}^{R-1} \frac{\mathbf{P}(\ln r_1)}{r_1^{\alpha_1-D+1}(R-r_1)^{\alpha-D}}, \end{aligned} \tag{B2}$$

where in the second line we extended the sum in $\vec{r}_2$ from $\overline{S}$ to $\overline{S_1}$ (which contains $\overline{S}$), in the third and the last lines we upper bounded the sums by integration [with a constant coefficient absorbed into $\mathbf{P}(\ln r_1)$], and the integrals can be calculated analytically due to the spherical geometry. Applying Lemma 2 to Eq. (B2), we obtain Eq. (5) with α_3 summarized in Tab. I.

Appendix C: Bounds for correlation decay: proof that $|\Omega_{XY}(iy)| \le |\bar{\Omega}_{XY}(0)|$

In this section we prove the claim we made in Sec. V that $|\Omega_{XY}(iy)|$ for $y \in \mathbb{R}$ can always be upper bounded by the upper bound of $|\Omega_{XY}(0)|$ obtained by Eq. (12). We prove this for the $\Omega_{XY}(\omega)$ in Sec. II C, i.e. power-law systems with $\alpha > D$, and the proofs for other cases are similar.

We begin by recalling a simple fact about subharmonic functions: if $p(\omega)$ is a real-valued subharmonic function and $q(\omega)$ is a real-valued harmonic function such that $p(\omega) \le q(\omega)$ on the boundary of a simply-connected domain S, then $p(\omega) \le q(\omega)$ everywhere in S. Now take $p(\omega)$ to be the subharmonic function $\ln|\Omega_{XY}(\omega)|$ and take $q(\omega)$ to be the unique harmonic function which agrees with $\ln\bar{\Omega}(r, y)$ on the boundary of the region S_b bounded by the parametric curve $g_v(be^{i\theta}), \theta \in [0, 2\pi]$, as plotted in Fig. 1, where $g_v(z)$ is defined in Eq. (18), $b \in (0, 1)$, and later we will consider the limit $b \to 1$. By construction, we have $p(\omega) \le q(\omega)$ on ∂S_b, therefore $p(\omega) \le q(\omega)$ everywhere in S_b. Using the mean-value property of harmonic functions, in the limit $b \to 1$ we have

$$\begin{aligned} \lim_{b\to 1} q(0) &= \lim_{b\to 1} \frac{1}{2\pi}\int_0^{2\pi} q[g_v(be^{i\theta})]d\theta \\ &= \lim_{b\to 1} \frac{1}{2\pi}\int_0^{2\pi} \ln\bar{\Omega}[r, |\mathrm{Im} g_v(be^{i\theta})|]d\theta \\ &= \ln|\bar{\Omega}_{XY}(0)|. \end{aligned} \tag{C1}$$

Therefore, to prove that $|\Omega_{XY}(iy)| \le |\bar{\Omega}_{XY}(0)|$, it suffices to prove that $q(iy)$ is monotonically decreasing in y for $y \ge 0$. In the following we prove this for any $b \in (0, 1)$.

Since $q(\omega)$ is harmonic, for illustrative purpose we use the language of electrostatics. From the expression of $\bar{\Omega}(r, y)$ in Eq. (17) it is clear that on the boundary of S_b the potential $q(\omega)$ is strictly decreasing in the direction of increasing $|y|$. In the following we use proof by contradiction: if $q(iy)$ is not monotonically in y for $y \ge 0$, then there must exist y_1, y_2 satisfying $0 < y_1 < y_2 < g_v(ib)/i$ such that $q(iy_1) = q(iy_2)$. Let l_1, l_2 be the equipotential lines passing through iy_1, iy_2, respectively. Equipotential lines cannot terminate in free space, since otherwise it would imply there is an electric charge at the end point. l_1 and l_2 cannot intersect anywhere, since for example if they intersect at a point $x + iy$ with $x > 0, y > 0$, then by symmetry they also intersect at $-x + iy$, which implies that l_1, l_2 enclose a region in which $q(\omega)$ is a constant, which is impossible for a non-constant harmonic function. By similar logic (and using the mirror symmetry with respect to the real axis) neither l_1 nor l_2 can intersect with the real axis, so we can focus our attention on the upper half plane. Furthermore, at most one of l_1, l_2 can intersect with the boundary of S_b, since $q(\omega)$ is strictly decreasing in the direction of increasing $|y|$ on the boundary. Without loss of generality suppose l_1 does not intersect the boundary. Then the only remaining possibility is that l_1 is a closed curve inside S_b. But this implies that $q(\omega)$ is constant in the interior of l_1, which is impossible for a non-constant harmonic function. In conclusion, $q(iy)$ must be monotonically decreasing in y for $y > 0$. [It is straightforward to rule out the possibility of $q(iy)$ being monotonically increasing in y: since if that's the case there must exist x, y with $0 < x < g_v(b), 0 < y < g_v(ib)/i$ such that $q(x) = q(iy)$. Considering the equipotential curve passing through x, iy, $-x$, and $-iy$, we reach a similar contradiction.]

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[51] The method in Ref. [9] does not use QAC. It is not clear to us if that method can be generalized to the power-law case, but when applied to short-range interacting systems, that method is more complicated than ours and gives looser bounds. Therefore we expect that even if that method can be generalized to the power-law case, the resulting bounds would be looser than our bounds listed in Tab. I.

[52] For fermionic systems, h_X is an operator that is even in the fermion creation/annihilation operators, and only involve fermionic modes inside the region X. This is enough to guarantee the important condition for the LRBs used later in this paper: for non-overlapping regions X_1, X_2 (i.e. $X_1 \cap X_2 = \emptyset$), we always have $[h_{X_1}, h_{X_2}] = 0$.

[53] Note that this definition of two-body interactions is more general than the more common definition–here we actually allow interaction between any two clusters of sites, where the radius of each cluster is no larger than a fixed number. It is easy to see that the results in Ref. [29] are still valid for two-cluster interactions, since one can always regroup lattice sites to make interactions two-body, as long as the cluster sizes have an upper bound.

[54] Although for the case $\alpha > 2D + 1$, a stronger linear light-cone has been obtained in Refs. [31, 32], they do not give us qualitatively tighter LPPL bounds. Indeed, the LRB in Ref. [31] gives $C(r,t) \leq t/r$ in 1D, leading to an LPPL bound with $\alpha_1 = 1$, which is worse than our current results, and the LRB in Ref. [32] can at most improve the subleading prefactor in Eq. (3), since $\alpha_1 = \alpha$ is already tight for generic systems.

[55] This conformal mapping is motivated by the answer to our question on math stackexchange https://math.stackexchange.com/questions/4443310/a-boundary-value-problem-of-a-harmonic-potential. We thank the user named "messenger" for providing the answer.

[56] Actually, we only need to assume that there exists a gapped path $\hat{H}(\lambda)$ that connects $\hat{H} - \hat{V}_Y$ and $\hat{H}$. We do not require $\hat{H}(\lambda)$ to be a linear interpolation, but $\hat{H}(\lambda)$ should only differ from $\hat{H}$ near the boundary.